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Invariant class operators in the decoherent histories analysis of timeless quantum theories

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The decoherent histories approach to quantum theory is applied to a class of reparametrization-invariant models whose state is an energy eigenstate. A key step in this approach is the construction of class operators characterizing the questions of physical interest, such as the probability of the system entering a given region of configuration space without regard to time. In nonrelativistic quantum mechanics these class operators are given by time-ordered products of projection operators. But in reparametrization-invariant models, where there is no time, the construction of the class operators is more complicated, the main difficulty being to find operators which commute with the Hamiltonian constraint (and so respect the invariance of the theory). Here, inspired by classical considerations, we put forward a proposal for the construction of such class operators for a class of reparametrization-invariant systems. They consist of continuous infinite temporal products of Heisenberg picture projection operators. We investigate the consequences of this proposal in a number of simple models and also compare with the evolving constants method. The formalism developed here is ultimately aimed at cosmological models described by a Wheeler-DeWitt equation, but the specific features of such models are left to future papers.

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I. INTRODUCTION

A. Opening remarks

A problem attracting some interest in recent years concerns the quantization of simple cosmological models which possess no intrinsic time parameter, and which are described by an equation of the Wheeler-DeWitt type

$$H\Psi = 0. \quad (1.1)$$

The absence of a time parameter together with the associated reparametrization invariance represents a particular challenge to conventional methods of quantization and interpretation and it has proved surprisingly difficult to extract probabilities from the wave function. Two particular approaches have made interesting progress in this area: the evolving constants method [1–7], and the decoherent histories approach [8–13]. The aim of this paper is to develop further the decoherent histories quantization of these “timeless” theories described by an equation of the form Eq. (1.1). The present paper will concentrate on the quantum-mechanical aspects of quantizing timeless models living in an energy eigenstate, leaving the specific features of cosmological models to future papers.

B. The decoherent histories approach

We first briefly review the decoherent histories approach in nonrelativistic quantum theory described by a Schrödinger equation [14–20]. In the decoherent histories approach probabilities are assigned to histories via the formula,

$$p(\alpha_1, \alpha_2, \dots) = \text{Tr}(C_{\underline{\alpha}} \rho C_{\underline{\alpha}}^\dagger), \quad (1.2)$$

where $C_{\underline{\alpha}}$ denotes a time-ordered string of projectors at times $t_1 \dots t_n$,

$$C_{\underline{\alpha}} = P_{\alpha_n}(t_n) \dots P_{\alpha_2}(t_2) P_{\alpha_1}(t_1) \quad (1.3)$$

and $\underline{\alpha}$ denotes the string $\alpha_1, \alpha_2, \dots, \alpha_n$. The projection operators are in the Heisenberg picture,

$$P_{\alpha_k}(t_k) = e^{iHt_k} P_{\alpha_k} e^{-iHt_k}, \quad (1.4)$$

where the projectors satisfy

$$\sum_{\alpha} P_{\alpha} = 1 \quad (1.5)$$

and

$$P_{\alpha} P_{\beta} = \delta_{\alpha\beta} P_{\alpha}. \quad (1.6)$$

We are interested in sets of histories which satisfy the condition of decoherence, which is that the decoherence functional

$$D(\underline{\alpha}, \underline{\alpha}') = \text{Tr}(C_{\underline{\alpha}} \rho C_{\underline{\alpha}'}^\dagger) \quad (1.7)$$

is zero when $\underline{\alpha} \neq \underline{\alpha}'$. Decoherence implies the weaker condition that $\text{Re} D(\underline{\alpha}, \underline{\alpha}') = 0$ for $\underline{\alpha} \neq \underline{\alpha}'$, and this is equivalent to the requirement that the above probabilities satisfy the probability sum rules. We normally work with the stronger condition of decoherence, which is related to the existence of records, corresponding to generalized measurements [15,21].

Now some simple observations relevant to what follows. The class operators Eq. (1.3) defined above satisfy

$$\sum_{\underline{\alpha}} C_{\underline{\alpha}} = 1 \quad (1.8)$$

However, in nonrelativistic quantum mechanics, one can equally well define the class operators to be

$$C_{\underline{\alpha}} = P_{\alpha_n} e^{-iH(t_n - t_{n-1})} P_{\alpha_{n-1}} \dots e^{-iH(t_2 - t_1)} P_{\alpha_1} \quad (1.9)$$

as long as the initial density matrix is redefined to absorb a unitary evolution factor (the unitary factors at the final time cancel out in the decoherence functional). This alternative class operator satisfies

$$\sum_{\alpha} C_{\alpha} = e^{-iH(t_n - t_1)}. \quad (1.10)$$

The distinction between these two class operators is trivial in nonrelativistic quantum mechanics but not so in reparametrization-invariant theories where one has to ask afresh what a class operator actually is. The definition Eq. (1.9) with the property Eq. (1.10) views the class operator as the decomposition of a propagator, and is best thought of in terms of a restricted sum over paths in a path integral. The definition Eq. (1.3) with the property Eq. (1.8), on the other hand, views a class operator as the generalization of a projection operator since clearly it would be a projection operator if all the projections at different times commute. The difference between these two views is irrelevant in nonrelativistic quantum mechanics but can have a significant influence when it comes to generalizations to reparametrization-invariant theories, as we shall see shortly.

C. Decoherent histories for systems without time

The structure of the decoherent histories approach is very general and readily applies to a wide variety of situations, provided one specifies a number of things in the construction of the decoherence functional Eq. (1.7), such as the inner product structure and the form of the class operators. Here we are concerned with reparametrization-invariant theories which are characterized by a constraint equation of the form

$$H|\Psi\rangle = 0, \quad (1.11)$$

where H is usually quadratic in all the momenta. Important examples are the Klein-Gordon equation of relativistic quantum mechanics and the Wheeler-DeWitt equation of quantum cosmology. The solutions to this equation are usually not normalizable in the usual Schrödinger inner product, so we use instead the so-called induced (or Rieffel) inner product [22]. This involves first considering eigenstates of H ,

$$H|\Psi_{Ek}\rangle = E|\Psi_{Ek}\rangle \quad (1.12)$$

where k is a degeneracy label. The spectrum of H is typically continuous and in the usual inner product we have

$$\langle\Psi_{Ek}|\Psi_{E'k'}\rangle = \delta(E - E')\delta(k - k') \quad (1.13)$$

The induced inner product between the eigenstates with the same E (including $E = 0$) is then defined, loosely speaking, by dropping the δ function in E , that is

$$\langle\Psi_{Ek}|\Psi_{E'k'}\rangle_I = \delta(k - k') \quad (1.14)$$

In practical terms, this means working with the usual inner

product, regularizing all expressions by working with eigenstates of H and then dropping $\delta(E - E')$ at the end. See Refs. [9–11] for applications similar to those considered here.

We are interested in the construction of the class operators for systems of this type. The key property of reparametrization-invariant theories is that they generally do not possess a variable to play the role of time, hence all questions that one asks about the system must not refer to time in any way. We will concentrate on the following useful question: given that the system's state satisfies the constraint equation, what is the probability of finding the system in a region Δ of configuration space, without regard to time? The question is clearly a sensible one classically, since the system has a number of classical trajectories and one can ask what proportion of them pass through the region in question. Moreover, classically, it is also a reparametrization-invariant question, since an entire classical trajectory is a reparametrization-invariant object [11]. To answer this question in the quantum case we need to find a suitable class operator.

It is generally held that for reparametrization-invariant theories, the most significant class of physical questions involve operators which commute with the Hamiltonian [1–3, 23–25]. These are referred to as “observables” and are the analogues of gauge-invariant quantities in gauge theories. This issue is not without debate and subtlety in the case of theories invariant under reparametrizations [26], but in this paper we will go along with this general idea. We therefore seek a class operator C_{Δ} satisfying

$$[H, C_{\Delta}] = 0 \quad (1.15)$$

and which corresponds to the statement that the system passes through the region Δ without reference to time. With reference to the discussion of Sec. I B, we adopt the projection operator view of the class operator, so the C_{Δ} becomes the identity when Δ becomes the entire configuration space and is also a projector when everything commutes.

On the other hand, if we looked instead for a class operator which is the analogue of the propagator form Eq. (1.9), then C_{Δ} become $\delta(H)$ when Δ becomes the entire configuration space (although this is essentially the identity when operating on solutions to the constraint equation). We then expect that class operator to satisfy the constraint equation

$$HC_{\Delta} = 0. \quad (1.16)$$

However, this propagator viewpoint naturally leads to a path integral construction which, in earlier works, was found difficult to reconcile with the constraint equation, Eq. (1.16) [10, 11]. In this paper we will therefore concentrate on the projection operator form, the generalization of Eq. (1.3), and reparametrization invariance is easily maintained.

D. This paper

In Sec. II we describe the earlier attempts to construct class operators for timeless models and the associated mathematical machinery that we will need here. In Sec. III we describe our proposal for new class operators which are compatible with the constraint. They consist of infinite products in time of projection operators in the Heisenberg picture. We compare the decoherent histories approach with the “evolving constants” method in Sec. IV. In Sec. V, we show that the decoherent histories approach together with the new class operators gives sensible and expected results for the nonrelativistic particle in parametrized form. In Sec. VI we compute the class operators and decoherence functional for a simple one-dimensional example and we apply this understanding to the relativistic particle in Sec. VII. We look at a simple example in two dimensions in Sec. VIII. The very different and simpler case of systems of harmonic oscillators is covered in Sec. IX. We summarize and conclude in Sec. X.

II. BACKGROUND

We begin by describing the propagator viewpoint for the construction of class operators for timeless systems. As stated, this has difficulties in relation to the constraint, but the details of the construction are important. Recall that we are interested in the question, given that the system is in an energy eigenstate, what is the probability of finding the particle in a region Δ of configuration space, without regard to time?

We will consider a system whose d -dimensional configuration space is \mathbb{R}^d and it will generally be useful to denote their coordinates by a vector \mathbf{x} , although when talking about the relativistic particle, we will use the usual notations x or x^μ . The propagator approach to defining the class operators is to define them by summing over all paths in the configuration space between given endpoints which pass through the region Δ [8–11]. In this approach, the class operator of interest is therefore given by

$$C_\Delta(\mathbf{x}'', \mathbf{x}') = \int_{-\infty}^{\infty} dT g_\Delta(\mathbf{x}'', T|\mathbf{x}', 0) \quad (2.1)$$

The integrand is given by a standard path integral (of nonrelativistic type)

$$g_\Delta(\mathbf{x}'', T|\mathbf{x}', 0) = \int \mathcal{D}\mathbf{x} \exp(iS[\mathbf{x}(t)]) \quad (2.2)$$

where the sum is over all paths from \mathbf{x}' to \mathbf{x}'' in time T which pass through Δ and $S[\mathbf{x}(t)]$ is an action of the usual form

$$S[\mathbf{x}(t)] = \int_0^T dt (f_{ij} \dot{x}_i \dot{x}_j - V(\mathbf{x})) \quad (2.3)$$

for some metric on the configuration space f_{ij} (whose explicit form will be unimportant in this section). This definition seems reasonable since it is an obvious general-

ization of Eqs. (1.9) and (1.10). Also, if we let the region Δ become the whole configuration space, then g is a solution to the Schrödinger equation, and, since the integration range of T is infinite, $C(\mathbf{x}'', \mathbf{x}')$ is a solution to the constraint equation,

$$HC = 0. \quad (2.4)$$

(If we are interested in an eigenstate of energy E , then we may assume that E has been absorbed into the potential V). However, there is a fundamental problem with this construction, which is that the class operator does not appear to satisfy the constraint equation everywhere, except for the case when Δ is the whole configuration space.

To see this, it is necessary to go into more detail about the construction of the above class operators. These details will also be important for the construction of projector-type class operators which commute with the constraint. We first introduce the (provisional) class operator for not entering the region Δ , which is given by a restricted sum over paths that do not enter Δ ,

$$C_r(\mathbf{x}'', \mathbf{x}') = \int_{-\infty}^{\infty} dT g_r(\mathbf{x}'', T|\mathbf{x}', 0) \quad (2.5)$$

Here $g_r(\mathbf{x}'', T|\mathbf{x}', 0)$ is the nonrelativistic restricted propagator, defined by a sum over paths in fixed time T that do not enter Δ . It vanishes when either endpoint is in Δ or on its boundary. Since the set of all paths between the fixed endpoints either pass through Δ or not, we have

$$g(\mathbf{x}'', T|\mathbf{x}', 0) = g_r(\mathbf{x}'', T|\mathbf{x}', 0) + g_\Delta(\mathbf{x}'', T|\mathbf{x}', 0) \quad (2.6)$$

and correspondingly

$$C(\mathbf{x}'', \mathbf{x}') = C_r(\mathbf{x}'', \mathbf{x}') + C_\Delta(\mathbf{x}'', \mathbf{x}'). \quad (2.7)$$

Here, $C(\mathbf{x}'', \mathbf{x}')$ denotes the sum over all paths, and in fact

$$C(\mathbf{x}'', \mathbf{x}') = \langle \mathbf{x}'' | \delta(H) | \mathbf{x}' \rangle. \quad (2.8)$$

There is a way of writing the restricted propagator which will be useful for later sections. We introduce the projection operator P onto the region Δ ,

$$P = \int_{\Delta} d^d x |\mathbf{x}\rangle \langle \mathbf{x}| \quad (2.9)$$

together with the complementary projector $\bar{P} = 1 - P$ onto the region $\bar{\Delta}$ outside Δ . Suppose we divide the time interval $[t', t'']$ into discrete points, $t' = t_0 < t_1 < t_2 < \dots < t_{n-1} < t_n = t''$, where $t_{k+1} - t_k = \delta t$. We introduce the operator version $g_r(t'', t')$ of the restricted propagator, $g_r(\mathbf{x}'', t''|\mathbf{x}', t')$, so

$$g_r(\mathbf{x}'', t''|\mathbf{x}', t') = \langle \mathbf{x}'' | g_r(t'', t') | \mathbf{x}' \rangle. \quad (2.10)$$

The operator version is then given by

$$g_r(t'', t') = \lim_{\delta t \rightarrow 0} \bar{P} e^{-iH(t_n - t_{n-1})} \bar{P} e^{-iH(t_{n-1} - t_{n-2})} \dots \times \bar{P} e^{-iH(t_1 - t_0)} \bar{P} \quad (2.11)$$

where the limit is $\delta t \rightarrow 0$, $n \rightarrow \infty$ with $n\delta t = (t'' - t')$ held constant. From this one can clearly see that $g_r(\mathbf{x}'', t'' | \mathbf{x}', t')$ vanishes if either endpoint is in Δ . One may also see that it does not quite satisfy the Schrödinger equation, but satisfies instead,

$$\left(i \frac{\partial}{\partial t''} - H\right) g_r(t'', t') = [\bar{P}, H] g_r(t'', t'). \quad (2.12)$$

Because \bar{P} is a projection operator, in the \mathbf{x} representation the right-hand side consists only of δ -functions on the boundary of Δ . So the restricted propagator almost satisfies the Schrödinger equation, but just fails at the boundary. Correspondingly, when used to construct the class operator $C_r(\mathbf{x}'', \mathbf{x}')$ in Eq. (2.5), it fails to satisfy the constraint because of δ -functions on the boundary. Consequently, C_Δ also fails to satisfy the constraint, because of Eq. (2.7).

It should be noted that constructions such as Eq. (2.5) can, in fact, be argued to be reparametrization invariant and one might therefore expect that it satisfies the constraint equation. The fact that it does not quite satisfy the constraint is related to subtle differences between the way reparametrizations act in configuration space versus phase space [26].

Another useful formula for the construction of these class operators is the so-called path decomposition expansion [27–29]. The propagator $g_\Delta(\mathbf{x}'', t'' | \mathbf{x}', t')$ is given by a sum over paths which enter the region Δ . These paths may be partitioned according to the time t_c and place \mathbf{x}_c at which they cross the boundary Σ of Δ for the first time. The crossing propagator may then be written,

$$g_\Delta(\mathbf{x}'', t'' | \mathbf{x}', t') = \int_{t'}^{t''} dt_c \int_{\Sigma} d^{d-1} \mathbf{x}_c g(\mathbf{x}'', t'' | \mathbf{x}_c, t_c) \frac{i}{2m} \times \mathbf{n} \cdot \nabla g_r(\mathbf{x}_c, t_c | \mathbf{x}', t') \quad (2.13)$$

where the normal \mathbf{n} points towards the restricted propagation region. Although the restricted propagator g_r vanishes on Σ , its normal derivative does not (if defined by first taking the derivative and then letting \mathbf{x}_c approach the surface from within the restricted propagation region). In fact the combination $(i/2m) \mathbf{n} \cdot \nabla g_r(\mathbf{x}_c, t_c | \mathbf{x}', t')$ represents a sum over paths which do not cross Σ but end on it.

The path decomposition expansion was used in Ref. [10] to compute class operators corresponding to crossings of a spacelike surface in relativistic quantum mechanics. As stated above, the class operators constructed using the above methods failed to satisfy the constraint. However, following a suggestion in Ref. [9], it was shown that operators satisfying the constraint and yielding sensible results could be obtained by some simple and physically reasonable modifications. But this procedure is rather *ad hoc* and it was not clear how to turn it into a general definition of the class operator.

Another clue as to how class operators should be constructed was found in Ref. [11], which considered general minisuperspace models and attempted to construct class

operators for them. The starting point was the construction of probabilities for timeless coarse grainings in the classical theory. Suppose we have a classical theory described by a phase space probability distribution function $w(\mathbf{p}, \mathbf{x})$ satisfying

$$\{H, w\} = 0 \quad (2.14)$$

(the classical analogue of the constraint equation). Let $f_\Delta(\mathbf{x})$ denote the characteristic function of the region Δ , so is 1 or 0, depending on whether \mathbf{x} is inside or outside Δ . Now introduce the classical solution $\mathbf{x}^{cl}(t)$ passing through the phase space point \mathbf{p}, \mathbf{x} . Then the quantity

$$\tau_\Delta = \int_{-\infty}^{\infty} dt f_\Delta(\mathbf{x}^{cl}(t)) \quad (2.15)$$

is the amount of parameter time spent by the trajectory in the region Δ . This quantity has the important property that it has vanishing Poisson bracket with the Hamiltonian,

$$\{H, \tau_\Delta\} = 0 \quad (2.16)$$

so is a classical observable. To determine whether or not the trajectory passes through Δ , we only need to know if τ_Δ is positive or zero. It follows that the probability of entering Δ is given by

$$p_\Delta = \int d^d p d^d x w(\mathbf{p}, \mathbf{x}) \theta\left(\int_{-\infty}^{\infty} dt f_\Delta(\mathbf{x}^{cl}(t)) - \epsilon\right). \quad (2.17)$$

Here, ϵ is a small parameter which goes to zero through positive values, and is included to avoid ambiguities in the θ -function at zero argument. The whole expression is invariant under reparametrizations, since each part of it is. Similarly, the probability for not entering the region is obtained by flipping the sign of the argument in the θ -function. (Note that there is an issue of normalization of $w(\mathbf{p}, \mathbf{x})$ in Eq. (2.17), since $w(\mathbf{p}, \mathbf{x})$ is constant along the classical trajectories. This issue is in fact resolved by the normalization in the analogous quantum case, as discussed in Ref. [11]).

Inspired by the classical case, it was suggested in Ref. [11] that in the quantum case, the class operator in the semiclassical approximation is given by

$$C_\Delta(\mathbf{x}_f, \mathbf{x}_0) = \theta\left(\int_{-\infty}^{\infty} dt f_\Delta(\mathbf{x}_0^f(t)) - \epsilon\right) B(\mathbf{x}_f, \mathbf{x}_0) e^{iA(\mathbf{x}_f, \mathbf{x}_0)} \quad (2.18)$$

where $B e^{iA}$ is the usual unrestricted semiclassical propagator, and $\mathbf{x}_0^f(t)$ denotes the classical path connecting \mathbf{x}_0 to \mathbf{x}_f . This object satisfies the constraint in the semiclassical approximation and gave sensible results, but no fully quantum version was given.

Note that the classical and semiclassical results Eqs. (2.17) and (2.18) involve entire classical trajectories, not trajectories of finite length between fixed endpoints as indicated by constructions such as Eq. (2.1). This is significant since, as argued previously, a whole classical

trajectory is reparametrization invariant, whereas a section of classical trajectory is not [11]. Hence one of the key ideas in the quantum theory is to get away from propagation between fixed endpoints and towards objects which capture the idea of an entire trajectory, as we will see shortly.

III. NEW CLASS OPERATORS

Given the above background and difficulties, the question now is how to define class operators that commute with the constraint and that give sensible semiclassical results. In this section we will focus on the case in which the unphysical parameter time t takes an infinite range. The special case in which the parameter time is periodic will be treated in Sec. IX.

A useful hint towards constructing class operators comes from the θ -function used in the expressions (2.17) and (2.18). Suppose we are interested in the probability of not entering Δ . Let us write the appropriate θ -function in terms of its Fourier transform,

$$\theta(\epsilon - \tau_\Delta) = \int \frac{dk}{ik} e^{ik(\epsilon - \tau_\Delta)} \quad (3.1)$$

where τ_Δ is given by Eq. (2.15). As stated, this object is reparametrization-invariant in that it has vanishing Poisson bracket with H . Now consider a discretized version of the time integral for τ_Δ , so we split the time into small intervals of size δt , and we have

$$\tau_\Delta \approx \delta t \sum_{n=-\infty}^{\infty} f_\Delta(\mathbf{x}(t_n)) \quad (3.2)$$

with exact agreement with the original expression in the limit $\delta t \rightarrow 0$. This then means that the θ -function is given by the continuum limit of the expression

$$\theta(\epsilon - \tau_\Delta) = \int \frac{dk}{ik} e^{ik\epsilon} \prod_{n=-\infty}^{\infty} \exp(-ik\delta t f_\Delta(t_n)). \quad (3.3)$$

But now f_Δ is a characteristic function, so is 0 or 1. It follows that

$$\exp(-ik\delta t f_\Delta) = f_{\bar{\Delta}} + e^{-ik\delta t} f_\Delta \quad (3.4)$$

where $f_{\bar{\Delta}} = 1 - f_\Delta$ is the characteristic function for the region $\bar{\Delta}$ outside Δ , and therefore

$$\theta(\epsilon - \tau_\Delta) = \int \frac{dk}{ik} e^{ik\epsilon} \prod_{n=-\infty}^{\infty} [f_{\bar{\Delta}}(t_n) + e^{-ik\delta t} f_\Delta(t_n)]. \quad (3.5)$$

This result has a very appealing form. When the product is expanded out, we get sums of products of the characteristic functions f_Δ and $f_{\bar{\Delta}}$, so the first term, for example, is the continuum limit of

$$\prod_{n=-\infty}^{\infty} f_{\bar{\Delta}}(t_n) \quad (3.6)$$

This quantity is clearly equal to 1 for a classical history in which the particle is outside Δ at every point along its trajectory and is zero otherwise. The other terms involve similar histories including the function f_Δ , so these are histories which enter Δ for some of the time. The integration over k produces a θ -function ensuring that only histories which spend time less than ϵ in the region Δ are included. In particular, as $\epsilon \rightarrow 0$, the only term that is left is the first term, Eq. (3.6). This is reparametrization-invariant because the expression it was derived from is. The important conclusion from this is that we might therefore expect to obtain reparametrization-invariant class operators in the quantum theory by taking infinite products of projection operators.

Turning now to the quantum theory, it is well known that operators commuting with H can be constructed using the formula,

$$A = \int_{-\infty}^{\infty} dt B(t) \quad (3.7)$$

where

$$B(t) = e^{iHt} B e^{-iHt}. \quad (3.8)$$

(See Ref. [3] for example). Suppose we let $B = \ln b$. Then, very loosely speaking

$$A = \int_{-\infty}^{\infty} dt B(t) = \ln \left(\prod_{t=-\infty}^{\infty} b(t) \right). \quad (3.9)$$

That is, to the extent that the continuous product over t is defined, we expect that operators of the form

$$\prod_{t=-\infty}^{\infty} b(t) \quad (3.10)$$

will commute with H .

Given these motivational remarks, we now give the new proposal for class operators for trajectories that never enter the region Δ . As before, denote by P the projector onto Δ and \bar{P} the projector onto the outside of Δ . Then our proposal for the class operator for trajectories not entering Δ is the time-ordered infinite product,

$$C_{\bar{\Delta}} = \prod_{t=-\infty}^{\infty} \bar{P}(t). \quad (3.11)$$

To define this more precisely, we first consider the product of projectors at a discrete set of times, $t' = t_0 < t_1 < t_2 < \dots < t_{n-1} < t_n = t''$, where $t_{k+1} - t_k = \delta t$. We define the intermediate quantity, $C_{\bar{\Delta}}(t'', t')$ as the continuum limit of the product of projectors,

$$C_{\bar{\Delta}}(t'', t') = \lim_{\delta t \rightarrow 0} \bar{P}(t_n) \dots \bar{P}(t_1) \bar{P}(t_0) \quad (3.12)$$

where the limit is $n \rightarrow \infty$, $\delta t \rightarrow 0$ with $t'' - t'$ fixed. Finally, the desired class operator is

$$C_{\bar{\Delta}} = \lim_{t'' \rightarrow \infty, t' \rightarrow -\infty} C_{\bar{\Delta}}(t'', t'). \quad (3.13)$$

This new class operator is clearly closely related to the restricted propagator defined above, Eq. (2.11) but differs by the presence of unitary evolution operators at either end. In particular, we have

$$C_{\bar{\Delta}}(t'', t') = e^{iHt''} g_r(t'', t') e^{-iHt'} \quad (3.14)$$

and therefore

$$C_{\bar{\Delta}} = \lim_{t'' \rightarrow \infty, t' \rightarrow -\infty} e^{iHt''} g_r(t'', t') e^{-iHt'}, \quad (3.15)$$

which is the most useful form of the class operator. This is the main result of this section.

As required, the new class operator commutes with H . This is implied by the construction, but more explicitly, we have from Eq. (3.14)

$$e^{iHs} C_{\bar{\Delta}}(t'', t') e^{-iHs} = e^{iH(t''+s)} g_r(t'', t') e^{-iH(t'+s)}. \quad (3.16)$$

This becomes independent of s as $t'' \rightarrow \infty$, $t' \rightarrow -\infty$, hence

$$[H, C_{\bar{\Delta}}] = 0. \quad (3.17)$$

Note that there is no reason at this stage why one should not use a different operator ordering of the projectors in Eq. (3.12). (This issue will become significant in the bound case treated later). Here, we investigate the consequences of the chosen ordering, which appears to be the simplest, but keeping in mind that a different choice may be appropriate.

The class operator Eq. (3.15) is quite different from the original proposal for this class operator, Eq. (2.5), in that it does not involve an integral over parameter time. Furthermore, unlike Eq. (2.5), the new class operator $C_{\bar{\Delta}}(\mathbf{x}'', \mathbf{x}')$ defined in this way does *not* in general vanish when either endpoint is in Δ , so it is not perfectly localized in $\bar{\Delta}$. In some sense, it corresponds to paths which do not enter the region Δ but are allowed to enter it at infinite parameter time. On the other hand, the new class operator is thoroughly compatible with the constraint equation, since it commutes with H , whereas Eq. (2.5) does not quite satisfy the constraint.

Generally, in the quantum theory, because the position operator does not commute with H , there is an incompatibility between localization in configuration space and the constraint equation. It is therefore necessary to make a choice as to which of these two requirements should be given precedence. The original proposal Eq. (2.5) has exact spatial localization, but is not fully compatible with the constraint. The new class operators are fully compatible with the constraint but are not perfectly localized in configuration space. Hence the current approach gives precedence to the constraint equation over localization.

Also, as noted earlier, the symmetry of reparametrization invariance is quite subtle in that Eq. (2.5) can be argued to be invariant under the configuration space form of reparametrizations, even though it is not fully compatible with the constraint. The symmetry generated by H is slightly larger than the configuration space form of reparametrizations, so in the new class operators we are demanding a slightly more restrictive notion of invariance than in Eq. (2.5). It would be of interest to explore these subtle differences in greater detail.

The class operator C_{Δ} for entering the region Δ is now simply defined by

$$C_{\Delta} = 1 - C_{\bar{\Delta}} \quad (3.18)$$

A more enlightening formula for it may however be obtained using the path decomposition expansion, Eq. (2.13). In particular, we clearly have

$$C_{\Delta} = \lim_{t'' \rightarrow \infty, t' \rightarrow -\infty} e^{iHt''} g_{\Delta}(t'', t') e^{-iHt'} \quad (3.19)$$

where $g_{\Delta}(t'', t')$ is defined by

$$g_{\Delta}(\mathbf{x}'', t'' | \mathbf{x}', t') = \langle \mathbf{x}'' | g_{\Delta}(t'', t') | \mathbf{x}' \rangle \quad (3.20)$$

and the left-hand side is given by the path decomposition expansion, Eq. (2.13).

It is not immediately clear from the definition of these class operators that they will exist in all situations of interest. In particular, one would expect that the continuous products over time and infinite limits will require careful attention. The proper mathematical framework for handling these quantities is the continuous tensor product structure defined by Isham *et al.* [30]. Here, we will proceed in a more informal way, and we will see by explicit computation in specific examples that the class operator exists and gives reasonable results. A more rigorous approach to quantizing models of this type is being pursued by Anastopoulos and Savvidou [13], using the structures developed by Isham *et al.* [30]. Future papers will address the connection between the present approach and these more rigorous approaches.

IV. COMPARISON WITH THE EVOLVING CONSTANTS METHOD

The decoherent histories approach considered here for timeless theories bears comparison with the evolving constants method of Rovelli [1] (for further developments see Refs. [3–7, 31, 32]). In that method, one constructs operators commuting with the constraint corresponding to physically interesting questions. For these operators, one may construct projections P_{α} onto ranges of the spectrum and the probabilities then have the usual form $\text{Tr}(P_{\alpha} \rho)$.

For example, suppose the system is a free particle in two dimensions, with Hamiltonian

$$H = \frac{p_1^2}{2m} + \frac{p_2^2}{2m}. \quad (4.1)$$

Suppose we are interested in the question, what is the value of x_1 when $x_2 = \tau$? The corresponding evolving constants variable is

$$X_1(\tau) = \int_{-\infty}^{\infty} ds x_1(s) \frac{dx_2(s)}{ds} \delta(x_2(s) - \tau) \quad (4.2)$$

where

$$x_i(s) = x_i + \frac{p_i}{m} s \quad (4.3)$$

for $i = 1, 2$. This clearly has vanishing Poisson bracket with H . Classically, the integral over s may be carried out with the result

$$X_1(\tau) = x_1 + \frac{p_1}{p_2}(\tau - x_2). \quad (4.4)$$

The $1/p_2$ factor presents difficulties in turning this into a self-adjoint operator, and as a consequence the spectrum of states is not orthonormal. One can still construct a positive operator-valued measure (POVM) onto a range of the spectrum but it will not be an exact projector, since it will not satisfy $P_\alpha P_\beta = 0$ for $\alpha \neq \beta$. This leads to a kind of imprecision in their definition. These POVMs however, may still be useful, in the same way the phase space localized quasiprojectors are useful.

To compare with the decoherent histories approach, suppose we are interested in the probability of entering or not entering a region Δ . Consider therefore the expression Eq. (2.15) for the parameter time spent in Δ , which in this simple model is

$$\tau_\Delta = \int_{-\infty}^{\infty} ds f_\Delta(x_1(s), x_2(s)). \quad (4.5)$$

Equation (4.5) may be written

$$\tau_\Delta = \int_{-\infty}^{\infty} ds \int_{\Delta} dy_1 dy_2 \delta(x_1(s) - y_1) \delta(x_2(s) - y_2). \quad (4.6)$$

The s integral may then be done with the result

$$\tau_\Delta = \int_{\Delta} dy_1 dy_2 \frac{m}{p_2} \delta\left(x_1 + \frac{p_1}{p_2}(y_2 - x_2) - y_1\right). \quad (4.7)$$

Importantly, the result depends only on the evolving constants operator $X_1(y_2)$ and on p_2 (both of which commute with H). Hence in the evolving constants approach one would consider the spectrum of the operator τ_Δ , using what is known about the spectrum of $X_1(y_2)$ and p_2 , and attempt to construct a projector or POVM onto ranges of the spectrum of τ_Δ (bearing in mind the difficulties noted above with self-adjointness). We can then find the probability of not entering Δ using the quasiprojector $\theta(\epsilon - \tau_\Delta)$.

In the decoherent histories approach, we also take Eq. (4.5) as the starting point. However, from that we

deduce the classical expression Eq. (3.6), which may be used for computation of the probability of not entering Δ . This is the starting point for the quantum theory, and, in particular, it inspires the construction of class operators in terms of products of projection operators, as described in the previous section. Importantly, class operators are *not* required to be self-adjoint operators, which in some sense means there is more freedom in the decoherent histories approach. On the other hand, in the decoherent histories approach probabilities cannot be defined in general, unless there is decoherence, so in this sense the theory is more restrictive than the evolving constants method.

The two resulting quantum theories are clearly quite different. However, what they have in common is that they take starting points which are classically equivalent. A more detailed comparison of these two approaches will be undertaken elsewhere.

V. THE PARAMETRIZED NONRELATIVISTIC PARTICLE

When developing a quantization scheme for parametrized systems, one of the most important simple systems to apply it to is the parametrized nonrelativistic particle. This is because its quantum theory is standard nonrelativistic quantum mechanics and it is therefore easy to check whether the expected results are reproduced by the methods described in Sec. III.

A. The parametrized particle and its quantization

The parametrized particle is the usual nonrelativistic particle but with the time coordinate t raised to the status of a dynamical variable, with conjugate momentum p_t . Its action in Hamiltonian form is

$$S = \int ds (p_x \dot{x} + p_t \dot{t} - NH) \quad (5.1)$$

where a dot denotes differentiation with respect to the unphysical time parameter s . (Note that t is physical time in this section). N is a Lagrange multiplier enforcing the constraint

$$H = p_t + h = 0 \quad (5.2)$$

where h is the usual Hamiltonian

$$h = \frac{p_x^2}{2m} + V(x). \quad (5.3)$$

Canonical quantization leads to the Schrödinger equation,

$$H|\psi\rangle = (p_t + h)|\psi\rangle = 0. \quad (5.4)$$

We are ultimately interested in solutions to the constraint equation, Eq. (5.4), which are normalized in terms of an inner product defined on spacetime (not just on space). Following the general scheme for constructing the induced inner product, we consider an enlarged

Hilbert space $\mathcal{H}_x \otimes \mathcal{H}_t$, where \mathcal{H}_x is the usual Hilbert space of wave functions $\psi(x)$. We may define states on this enlarged space of the form

$$|\Psi\rangle = \int dx dt |x\rangle \otimes |t\rangle \Psi(x, t) \quad (5.5)$$

where $|x\rangle$ and $|t\rangle$ are eigenstates of the operators \hat{x} and \hat{t} respectively. We then consider eigenstates of H ,

$$H|\Psi_\lambda\rangle = \lambda|\Psi_\lambda\rangle. \quad (5.6)$$

They are normalized using the auxiliary inner product defined on $\mathcal{H}_x \otimes \mathcal{H}_t$,

$$\langle\Psi_\lambda|\Psi'_{\lambda'}\rangle_A = \int dx dt \Psi_\lambda^*(x, t) \Psi'_{\lambda'}(x, t). \quad (5.7)$$

Since $H = p_t + h$, the solutions to the eigenvalue equation may be written

$$\Psi_\lambda(x, t) = \frac{1}{(2\pi)^{\frac{1}{2}}} e^{i\lambda t} \psi(x, t) \quad (5.8)$$

where $\psi(x, t)$ satisfies the Schrödinger equation, Eq. (5.4). It follows that

$$\langle\Psi_\lambda|\Psi'_{\lambda'}\rangle_A = \frac{1}{2\pi} \int dt \int dx e^{-i\lambda t + i\lambda' t} \psi^*(x, t) \psi'(x, t). \quad (5.9)$$

The integral contains within it the usual inner product

$$\langle\psi|\psi'\rangle_S = \int dx \psi^*(x, t) \psi'(x, t). \quad (5.10)$$

This has the important property that it is independent of time when the states obey the Schrödinger equation, so the time integral may be done in Eq. (5.9), pulling down a delta function $\delta(\lambda - \lambda')$. We thus obtain

$$\langle\Psi_\lambda|\Psi'_{\lambda'}\rangle_A = \delta(\lambda - \lambda') \langle\psi|\psi'\rangle_S. \quad (5.11)$$

This means that the expected Schrödinger inner product on surfaces of constant t is fully compatible with the induced inner product defined on the whole of spacetime.

We may now construct the decoherence functional for this system, for some interesting physical questions. For a pure initial state, the decoherence functional is

$$D(\alpha, \alpha') = \langle\Psi_{\lambda'}|C_{\alpha'}^\dagger C_\alpha|\Psi_\lambda\rangle. \quad (5.12)$$

It is constructed using the induced inner product and the class operators C_α must commute with the constraint, H . A useful formula for simplifying expressions of the form Eq. (5.12) (for the parametrized nonrelativistic particle only) is given in the appendix and will be used below.

B. Probabilities on surfaces of constant time

Now we consider the simple question, what is the probability of finding the particle in a range Δ of the x -axis at time t_0 ? We of course expect the standard answer

$$p_\Delta = \int_\Delta dx |\psi(x, t_0)|^2, \quad (5.13)$$

but it is important to see how this arises in a consistent quantization of the parametrized particle.

We assert that the appropriate class operator corresponding to this question is

$$C_{\Delta, t_0} = \int_{-\infty}^{\infty} ds \delta(\hat{t}(s) - t_0) f_\Delta(\hat{x}(s)), \quad (5.14)$$

where

$$\hat{t}(s) = e^{iHs} \hat{t} e^{-iHs} = t + s \quad (5.15)$$

$$\hat{x}(s) = e^{iHs} \hat{x} e^{-iHs} \quad (5.16)$$

and $f_\Delta(x)$ is a window function on the range Δ . The class operator clearly commutes with both H and \hat{t} . Classically, for the free particle, the class operator corresponds to the expression

$$C_{\Delta, t_0} = f_\Delta\left(x - \frac{p}{m}(t - t_0)\right) \quad (5.17)$$

which clearly has the right properties: it is equal to 1 for classical trajectories which cross $t = t_0$ in Δ and zero otherwise. Returning to the quantum case, one can see that

$$C_{\Delta, t_0} = \int_{-\infty}^{\infty} dt e^{-ih(t-t_0)} f_\Delta(\hat{x}) e^{ih(t-t_0)} \otimes |t\rangle\langle t|. \quad (5.18)$$

This is of the form Eq. (A2) from which we read off

$$B(0) = e^{iht_0} f_\Delta(\hat{x}) e^{-iht_0}. \quad (5.19)$$

Now a crucial simplification. Since f_Δ is a window function, $B(0)$ is in fact a projection operator, and therefore the class operator C_{Δ, t_0} is also a projection operator. This means that decoherence is automatic, between histories characterized by C_{Δ, t_0} and $1 - C_{\Delta, t_0}$ and we may immediately assign the probability

$$p_\Delta = \langle\Psi_{\lambda'}|C_{\Delta, t_0}|\Psi_\lambda\rangle. \quad (5.20)$$

Using Eq. (A6) and dropping the δ -function, this becomes

$$p_\Delta = \langle\psi|e^{iht_0} f_\Delta(\hat{x}) e^{-iht_0}|\psi\rangle \quad (5.21)$$

which agrees exactly with the expected result Eq. (5.13). Note also that since the class operator is a projection operator in this case, the decoherent histories analysis agrees exactly with the evolving constants approach.

C. Probabilities for spacetime regions

We now consider a more challenging question which is to consider probabilities for regions of spacetime. In particular, we pose the following question: given the initial wave function $\psi(x)$ at $t = 0$, what is the probability that the particle is found in the region $x < 0$ in the time interval $[0, \tau]$? This has been analyzed previously in the decoherent histories approach with the following results [33]. (See also

Refs. [34,35]). The decoherence functional is

$$D(\alpha, \alpha') = \langle \psi | g_{\alpha'}^\dagger g_\alpha | \psi \rangle \quad (5.22)$$

in the usual inner product. (Here we use g_α to denote class operators to avoid confusion with the class operators C_α defined on the enlarged space.) There are two class operators. First, there is the class operator for remaining in $x > 0$ during the time interval $[0, \tau]$, and this is the restricted propagator $g_r(\tau, 0)$. The other class operator may be written

$$g_c(\tau, 0) = e^{-ih\tau} - g_r(\tau, 0) \quad (5.23)$$

and corresponds to the situation in which the particle enters $x < 0$ at some time during $[0, \tau]$. (It may also be expressed in terms of the path decomposition expansion Eq. (2.13), but we will not need this here. Note also that these class operators reduce to the unitary operator $e^{-ih\tau}$ when the restrictions are removed). The histories are generally not decoherent, except for very special initial states, and the resultant probabilities are somewhat trivial [33]. However, our aim here is to show how the decoherence functional for this model is recovered from a quantization of the parametrized particle as a constrained system on an enlarged Hilbert space, in which its spacetime character is most transparent.

We take the decoherence functional Eq. (5.12) and seek a class operator commuting with $H = p_t + h$ corresponding to the statement that the particle never enters the region $x < 0$ during the time interval $[0, \tau]$. We denote this region Δ and we use $\bar{\Delta}$ to denote the region outside Δ . In the genuinely spacetime point of view used here, we may introduce projection operators onto the spacetime regions Δ and $\bar{\Delta}$. The projection onto Δ is

$$P = \theta(\tau - \hat{t})\theta(\hat{t})\theta(-\hat{x}) \quad (5.24)$$

and the projection onto $\bar{\Delta}$ is conveniently written

$$\bar{P} = \theta(-\hat{t}) + \theta(\tau - \hat{t})\theta(\hat{t})\theta(\hat{x}) + \theta(\hat{t} - \tau) \quad (5.25)$$

$$= \int_{-\infty}^{\infty} dt Y(t, \hat{x}) \otimes |t\rangle\langle t|, \quad (5.26)$$

where $Y(t, \hat{x})$ is an operator on \mathcal{H}_x , equal to $\theta(\hat{x})$ for $0 \leq t \leq \tau$ and equal to the identity otherwise. The class operator for remaining in $\bar{\Delta}$ (that is, never entering the region Δ) is of the form

$$C_r = \prod_{s=-\infty}^{\infty} \bar{P}(s). \quad (5.27)$$

This example is sufficiently simple that we can work directly with the infinite product (time-ordered) without encountering difficulties. We have, from Eq. (5.26),

$$\begin{aligned} C_r &= \int_{-\infty}^{\infty} dt \prod_{s=-\infty}^{\infty} Y(t + s, \hat{x}(s)) \otimes |t\rangle\langle t| \\ &= \int_{-\infty}^{\infty} dt \prod_{s=-t}^{\tau-t} \theta(\hat{x}(s)) \otimes |t\rangle\langle t|. \end{aligned} \quad (5.28)$$

From the definition of the restricted propagator, Eq. (2.11), we see that

$$C_r = \int_{-\infty}^{\infty} dt e^{ih(\tau-t)} g_r(\tau, 0) e^{iht} \otimes |t\rangle\langle t|. \quad (5.29)$$

This commutes with the constraint H and is of the desired form from Eq. (A2) with

$$B(0) = e^{ih\tau} g_r(\tau, 0). \quad (5.30)$$

We are also interested in the quantity

$$C_r^\dagger C_r = \int_{-\infty}^{\infty} dt e^{-iht} g_r(\tau, 0)^\dagger g_r(\tau, 0) e^{iht} \otimes |t\rangle\langle t| \quad (5.31)$$

which is also of the form Eq. (A2) with

$$B(0) = g_r(\tau, 0)^\dagger g_r(\tau, 0). \quad (5.32)$$

From these objects one can also construct the class operator for crossing,

$$C_c = 1 - C_r \quad (5.33)$$

and related objects such as $C_c^\dagger C_r$. It is now easy to see, using Eq. (A6), that we readily obtain the known form of the decoherence functional for this system. For example,

$$\langle \Psi_{\lambda'} | C_r^\dagger C_r | \Psi_\lambda \rangle = \delta(\lambda' - \lambda) \langle \psi | g_r(\tau, 0)^\dagger g_r(\tau, 0) | \psi \rangle \quad (5.34)$$

which, via the induced inner product prescription, agrees with the known result Eq. (5.22).

These results show that our proposal for class operators passes the important test of the quantization of the nonrelativistic particle in parametrized form. Furthermore, there is the added feature that it shows how spacetime questions in nonrelativistic quantum mechanics may be expressed in a genuinely spacetime form, since the decoherence functional and probabilities [such as the left-hand side of Eq. (5.34)] may be expressed in terms of an inner product and operators defined on spacetime.

VI. A SIMPLE ONE-DIMENSIONAL EXAMPLE

The parametrized nonrelativistic particle has the very special simplifying feature that the Hamiltonian is linear in one of the momenta. This is not the case in general. We therefore now consider some examples with a Hamiltonian quadratic in the momenta. We first consider a simple one-dimensional example involving the free particle. It is trivial in itself (except to show that the class operators can be easily calculated), but readily extends to higher dimension

and has important implications for the relativistic particle considered later.

A. Energy eigenstates

We first consider normalization of the energy eigenstates. We have

$$H|\psi\rangle = E|\psi\rangle, \quad (6.1)$$

where $H = p^2/2m$. For each E there are two solutions which are conveniently written,

$$\psi_E^\pm(x) = \left(\frac{m}{2E}\right)^{1/4} \frac{e^{\pm i|k|x}}{(2\pi)^{1/2}}, \quad (6.2)$$

where $|k| = \sqrt{2mE}$. In the usual inner product,

$$\langle\psi_1|\psi_2\rangle = \int dx \psi_1^*(x) \psi_2(x) \quad (6.3)$$

we have

$$\langle\psi_E^\pm|\psi_{E'}^\pm\rangle = \delta(E - E') \quad (6.4)$$

and

$$\langle\psi_E^\pm|\psi_{E'}^\mp\rangle = 0. \quad (6.5)$$

In the induced inner product prescription we therefore drop the δ -function and take the induced inner product between two eigenstates with the same E to be

$$\langle\psi_E^\pm|\psi_E^\pm\rangle_I = 1 \quad (6.6)$$

and

$$\langle\psi_E^\pm|\psi_E^\mp\rangle_I = 0. \quad (6.7)$$

B. Class operators for crossing or not crossing the origin

Given these preliminaries, we now consider the following simple question. Given that the system is in an energy eigenstate, what is the probability that the particle crosses or never crosses $x = 0$, irrespective of time? This is most easily handled by considering the class operator C_r describing the situation in which the particle is always in $x > 0$ or $x < 0$. The class operator C_c for crossing $x = 0$ is then given by

$$C_c = 1 - C_r. \quad (6.8)$$

Let P be the projector onto the positive x -axis,

$$P = \int_0^\infty dx |x\rangle\langle x| = \theta(\hat{x}) \quad (6.9)$$

(we use the hat symbol for operators only where clarity demands it). Then

$$\bar{P} = 1 - P = \theta(-\hat{x}) \quad (6.10)$$

is the projector onto the negative x -axis. The class operator C_r for remaining in $x > 0$ or $x < 0$ is then given by, in a

loose notation,

$$C_r = \prod_t P(t) + \prod_t \bar{P}(t) = C_r^+ + C_r^-, \quad (6.11)$$

where $+$ and $-$ denote the terms projecting onto the positive and negative x -axis, respectively. This expression is defined more formally in terms of restricted propagators, as in Eq. (3.15).

This situation is sufficiently simple for the method of images to work. The restricted propagator for the region $x > 0$ is therefore

$$g_r^+(x'', t|x', 0) = \theta(x'')\theta(x') [g(x'', t|x', 0) - g(x'', t|-x', 0)], \quad (6.12)$$

where

$$g(x'', t|x', 0) = \left(\frac{m}{2\pi it}\right)^{1/2} \exp\left(\frac{im}{2t}(x'' - x')^2\right) \quad (6.13)$$

is the free particle propagator. The restricted propagator may be usefully thought of as sum of two parts. The first term (in a semiclassical view) corresponds to the direct path from x' to x'' . The second term is usually thought of as propagation from the image point $-x'$. However, it may also be thought of as corresponding to the path which again starts at x' but is reflected off $x = 0$ before arriving at x'' . Differently put, the classical limit of a system described by a restricted propagator is one in which the Hamiltonian includes an infinite potential barrier at the boundary of the region in question, so its classical trajectories include paths which reflect off the boundary. These points will be important in the interpretation of the quantum results.

It is in fact quite useful to rewrite this in an operator form using the projection operators introduced above. We also introduce the reflection operator

$$R = \int_{-\infty}^\infty dx |x\rangle\langle -x| = \int_{-\infty}^\infty dp |p\rangle\langle -p| \quad (6.14)$$

and note that $[H, R] = 0$ and $R^2 = 1$. The restricted propagator in $x > 0$ may then be written in operator form as

$$g_r^+(t, 0) = P e^{-iHt} (1 - R) P. \quad (6.15)$$

From Eq. (3.15), the desired class operator C_r^+ is

$$C_r^+ = \lim_{t'' \rightarrow \infty, t' \rightarrow -\infty} P(t'')(1 - R)P(t'). \quad (6.16)$$

Now we need to take the infinite time limit in $P(t'')$ and $P(t')$. We have

$$P(t) = \theta(\hat{x}_t) = \theta\left(\hat{x} + \frac{\hat{p}t}{m}\right). \quad (6.17)$$

Naively, for very large positive or negative t the momentum term dominates, and we get

$$\lim_{t'' \rightarrow \infty} P(t'') = \theta(\hat{p}) \quad (6.18)$$

$$\lim_{t' \rightarrow -\infty} P(t') = \theta(-\hat{p}). \quad (6.19)$$

However, it is important to verify this result more carefully, by sandwiching it between an arbitrary pair of states. We have, using Eq. (6.13),

$$\begin{aligned} \langle \psi_1 | P(t) | \psi_2 \rangle &= \frac{m}{2\pi t} \int_0^\infty dx \int_{-\infty}^\infty dy_1 \int_{-\infty}^\infty dy_2 \psi_1^*(y_1) \psi_2(y_2) \\ &\times \exp\left(-\frac{im}{2t}(y_1 - x)^2 + \frac{im}{2t}(x - y_2)^2\right). \end{aligned} \quad (6.20)$$

Defining $p = mx/t$, this becomes

$$\begin{aligned} \langle \psi_1 | P(t) | \psi_2 \rangle &= \frac{1}{2\pi} \int_0^\infty dp \int_{-\infty}^\infty dy_1 \int_{-\infty}^\infty dy_2 \psi_1^*(y_1) \psi_2(y_2) \\ &\times \exp\left(ip(y_1 - y_2) - \frac{im}{2t}(y_1^2 - y_2^2)\right). \end{aligned} \quad (6.21)$$

Now, taking the limit $t \rightarrow \infty$, the last term in the exponent drops out, and the integrals over y_1 and y_2 produce the Fourier transformed wave functions, so we have

$$\lim_{t \rightarrow \infty} \langle \psi_1 | P(t) | \psi_2 \rangle = \int_0^\infty dp \tilde{\psi}_1^*(p) \tilde{\psi}_2(p) = \langle \psi_1 | \theta(\hat{p}) | \psi_2 \rangle. \quad (6.22)$$

This confirms the naive result Eq. (6.18).

We now have

$$C_r^+ = \theta(\hat{p})(1 - R)\theta(-\hat{p}) = -\theta(\hat{p})R\theta(-\hat{p}) \quad (6.23)$$

which may also be written

$$C_r^+ = -\theta(\hat{p})R = -R\theta(-\hat{p}). \quad (6.24)$$

Note that the first part of the restricted propagator (involving the direct path from initial to final point) drops out, leaving only the part corresponding to the reflected paths. In the momentum representation, this is

$$\langle k_2 | C_r^+ | k_1 \rangle = -\theta(k_2)\delta(k_2 + k_1). \quad (6.25)$$

Similarly, the class operator C_r^- is constructed using the restricted propagator in $x < 0$,

$$\begin{aligned} g_r^-(x'', t | x', 0) &= \theta(-x'')\theta(-x')[g(x'', t | x', 0) \\ &- g(x'', t | -x', 0)] \end{aligned} \quad (6.26)$$

and one readily finds that

$$C_r^- = -\theta(-\hat{p})R = -R\theta(\hat{p}). \quad (6.27)$$

Combining the + and - parts we have

$$C_r = -\theta(\hat{p})R - \theta(-\hat{p})R = -R, \quad (6.28)$$

a very simple expression. The class operator for crossing the surface is then

$$C_c = 1 - C_r = 1 + R. \quad (6.29)$$

As expected, all the above class operators, C_c , C_r , C_r^+ and C_r^- commute with H .

C. Decoherence functional and probabilities

We may now compute the decoherence functional and the probabilities. The off-diagonal part of the decoherence functional is

$$D(r, c) = \text{Tr}(C_r \rho C_c^\dagger) \quad (6.30)$$

computed in the induced inner product. We take a pure initial state which is an eigenstate of H . Noting that

$$C_c^\dagger C_r = -(1 + R)R = -(R + 1) \quad (6.31)$$

we have

$$D(r, c) = -\langle \Psi_E | (1 + R) | \Psi_E \rangle_I, \quad (6.32)$$

where $|\Psi_E\rangle$ is an energy eigenstate and the induced inner product is used. Now noting that these energy eigenstates may be written $|\Psi_E\rangle = \delta(H - E)|\phi\rangle$ for some fiducial state $|\phi\rangle$, and using the induced inner product Eq. (1.14), we find that there is decoherence only for states satisfying

$$\langle \phi | \delta(H - E)(1 + R) | \phi \rangle = 0. \quad (6.33)$$

(This expression is in fact real, so there is no difference between consistency and decoherence). The operator $(1 + R)$ produces the symmetric part of $|\phi\rangle$. The decoherence condition therefore implies that the fiducial wave function must be antisymmetric,

$$\phi(-x) = -\phi(x). \quad (6.34)$$

This means that $\Psi_E(x)$ must also be antisymmetric (using the fact that R commutes with H). For such wave functions, probabilities are defined and we get

$$p_r = \langle \Psi_E | C_r^\dagger C_r | \Psi_E \rangle_I = 1 \quad (6.35)$$

for the probability for not crossing since $C_r^\dagger C_r = R^2 = 1$. Similarly, the probability for crossing is zero.

It is useful to try and understand this result in terms of classical paths. Recall that first, only entire infinite classical paths are reparametrization invariant and second, that the restricted propagator corresponds to a classical situation in which there is an infinite barrier at $x = 0$. Classically, therefore, this result in some sense corresponds to classical trajectories which remain in $x > 0$ or $x < 0$ by bouncing off $x = 0$.

This result is not very physically enlightening but it is very similar to the result obtained in the decoherent histories analysis of the crossing time problem in nonrelativistic quantum mechanics. There, one looks for probabilities that, given an initial state, the particle will cross $x = 0$ during the time interval $[0, \tau]$. One finds that only antisymmetric wave functions give consistency and the crossing probability is zero [33]. More physically intuitive results in the crossing time problem are obtained—when

there is a decoherence mechanism in place [34]. We expect that to be the case here, too, but this will be explored in another paper [36].

D. A more detailed look at the crossing class operator

It is useful to give an alternative derivation of the crossing class operator Eq. (6.29) using the path decomposition expansion, Eq. (2.13). This is partly a consistency check but it will also give some insight into the form of the result.

We first consider the fixed time crossing propagator, taking into account the possibility of crossing the origin in either direction. Applying Eq. (2.13) together with the restricted propagators Eqs. (6.12) and (6.26), this is

$$\begin{aligned} g_c(x'', t''|x', t') &= \lim_{x \rightarrow 0^-} \int_{t'}^{t''} dt g(x'', t''|x, t) \frac{i}{2m} \\ &\quad \times \frac{\partial}{\partial x} g_r^+(x, t|x', t') \\ &\quad - \lim_{x \rightarrow 0^+} \int_{t'}^{t''} dt g(x'', t''|x, t) \frac{i}{2m} \\ &\quad \times \frac{\partial}{\partial x} g_r^-(x, t|x', t'), \end{aligned} \quad (6.36)$$

where the relative minus sign is because of the definition of the normal in Eq. (2.13). Now, using Eqs. (6.12) and (6.26), we have

$$\lim_{x \rightarrow 0^-} \frac{\partial}{\partial x} g_r^-(x, t|x', t') = 2\theta(-x') \frac{\partial}{\partial x} g(x, t|x', t')|_{x=0} \quad (6.37)$$

$$\lim_{x \rightarrow 0^+} \frac{\partial}{\partial x} g_r^+(x, t|x', t') = 2\theta(x') \frac{\partial}{\partial x} g(x, t|x', t')|_{x=0}. \quad (6.38)$$

We therefore have

$$\begin{aligned} g_c(x'', t''|x', t') &= \lim_{x \rightarrow 0} \int_{t'}^{t''} dt g(x'', t''|x, t) \frac{i}{m} \\ &\quad \times \frac{\partial}{\partial x} g(x, t|x', t') \epsilon(x'), \end{aligned} \quad (6.39)$$

where $\epsilon(x)$ is the signum function. This is conveniently written in operator form as

$$g_c(t'', t') = \frac{1}{m} \int_{t'}^{t''} dt e^{-iH(t''-t)} \delta(\hat{x}) \hat{p} e^{-iH(t-t')} \epsilon(-\hat{x}). \quad (6.40)$$

The desired crossing class operator is now given by

$$\begin{aligned} C_c &= \lim_{t'' \rightarrow \infty, t' \rightarrow -\infty} e^{+iHt''} g_c(t'', t') e^{-iHt'} \\ &= \frac{1}{m} \int_{-\infty}^{\infty} dt \delta(\hat{x}_t) |\hat{p}|, \end{aligned} \quad (6.41)$$

where we have used the fact that $\epsilon(-\hat{x}_t) \rightarrow \epsilon(\hat{p})$ as $t \rightarrow -\infty$ and $|\hat{p}| = \hat{p}\epsilon(\hat{p})$.

Equation (6.41) is the desired expression and shows very clearly that the class operator for crossing $x = 0$ involves some kind of flux at $x = 0$. Classically, it is easy to see that

this expression is equal to 1 for all classical paths, except those for which $p = 0$, in which case it is zero. Hence it clearly encapsulates the classical notion of surface crossing. One might argue that classical states with $p = 0$ are a set of measure zero so may be safely neglected. However, the $p = 0$ states seem to be crucial to understand the quantum case.

As an operator expression, Eq. (6.41) may be evaluated by sandwiching it between two momentum states:

$$\begin{aligned} \langle p'' | C_c | p' \rangle &= \frac{1}{m} \int_{-\infty}^{\infty} dt \exp\left(\frac{it}{2m}(p'^2 - p'^2)\right) \\ &\quad \times \langle p'' | \delta(\hat{x}) | p' \rangle | p' | \\ &= 2 | p' | \delta(p''^2 - p'^2) \\ &= \delta(p'' - p') + \delta(p'' + p'). \end{aligned} \quad (6.42)$$

It follows that

$$C_c = 1 + R \quad (6.43)$$

the expected result.

How are we to understand this result? Classically, we noted that all trajectories with $p \neq 0$ cross the origin. The result Eq. (6.43) ought therefore to be the quantum implementation of this idea. The key thing is that the operator $1 + R$ is zero when acting on states which are antisymmetric in x about the origin. Such states are also antisymmetric in p so clearly vanish at $p = 0$. Hence the crossing class operator annihilates a class of states with $p = 0$ and in this sense implements the classical notion of crossing.

Of course, there are many inequivalent ways to turn classical expressions into quantum operators, and one could imagine that a quantization procedure may exist which consistently drops the $p = 0$ states before quantization. This would avoid the difficulties of interpretation with reflected paths encountered earlier. However, this does not appear to be the case in the present quantization method.

We also remark that, as noted earlier, the evolving constants method also encounters difficulties with $p = 0$ states because of the $1/p$ factors arising in the evolving constants operators.

VII. THE RELATIVISTIC PARTICLE

The result of the previous section is readily extended to the relativistic particle in $3 + 1$ dimensions described by the constraint equation

$$H|\psi\rangle = (p_0^2 - \mathbf{p}^2)|\psi\rangle = 0. \quad (7.1)$$

We consider this in order to compare with a previous attempt to define class operators compatible with the constraint [9,10].

Consider the following question. What is the probability that a free relativistic particle never crosses the spacelike surface $x^0 = 0$? Classically, this probability must be zero, because every (timelike) classical trajectory crosses any

spacelike surface somewhere, since the classical trajectories are just straight lines. In Ref. [10], a (somewhat *ad hoc*) proposal was made to define class operators compatible with the constraint which coincide with this classical intuition. A quantum-mechanical probability of zero was thus obtained for not crossing the surface.

The question is readily addressed in the present approach using an elementary extension of the results of the one-dimensional model in the previous section. The class operator for not crossing $x^0 = 0$ is of the form Eq. (6.11) where the Hamiltonian is as in Eq. (7.1) and the projectors P and \bar{P} project onto the regions $x^0 > 0$ and $x^0 < 0$, respectively. One easily finds that decoherence is only possible for states antisymmetric about $x^0 = 0$ and the probability for not crossing is equal to 1. It is therefore the exactly opposite result to that obtained in Ref. [10].

What is the origin of the difference in results and which is the “correct” one? The key point is that in the present approach, the restricted propagators involved in the construction of the class operators, such as Eq. (6.12) involve two types of paths in a semiclassical picture, the direct paths and the reflected paths. As noted above [after Eq. (6.24)], the part of the propagators corresponding to the direct paths drops out, so the present approach consists entirely of the contribution from the reflected paths. The classical intuition of Ref. [10] was tacitly based on the direct paths, so the result is completely different. Since the reflected paths capture the important notion of the reflection of wave packets from a barrier, it is appropriate to take the present approach as the definitive one if we are to stay true to quantum-mechanical ideas.

In the closely related context of the arrival time problem, it has however been shown that, in the presence of a decohering environment, the effect of the reflected paths becomes less significant, and this is how classical intuition may become restored [34]. See also Ref. [37] for further relevant considerations of the relativistic particle.

This brief discussion of the relativistic particle indicates that the present proposal for class operators is not in fact a more developed statement of the *ad hoc* approach of Refs. [9–11], but a different proposal altogether.

VIII. THE FREE PARTICLE IN TWO DIMENSIONS

In two dimensions more interesting questions are possible. However, the general difficulty one expects to encounter for most questions is decoherence. In the absence of an environment, most situations will not have decoherent sets of histories. In the one-dimensional example, the lack of decoherence for general states is largely due to the feature of reflection at the boundaries of the regions of interest. It is therefore of interest to consider situations where this reflection will not happen.

In two (and more) dimensions, we have the possibility of eigenstates of H which are rotationally symmetric about the origin and may be thought of as superpositions of wave

packets moving radially. It therefore seems plausible that if we consider questions concerning regions whose boundaries lie along radial lines, there will be little or no possibility of crossing or reflection (since the wave function has no flux across the boundary). For example, given a rotationally symmetric state, we could ask for the probability that the particle is found in a wedge-shaped region emanating from the origin. This question bears some resemblance to Mott’s calculation of alpha-ray tracks [38]. He showed, using a series of model detectors, why an outgoing spherical wave produces a straight line track in a detector.

We therefore consider the case of a region Δ consisting of the wedge lying in the region $0 \leq \phi \leq \beta$ (in polar coordinates r, ϕ). We ask for the probability that the particle is always in the region Δ .

Following the general scheme, we require first the time-dependent propagator for the wedge region. For simplicity we restrict to the case where the angle β is $\beta = \pi/b$, where b is an integer. We also take b to be even (which turns out to be simplest to deal with). Then the restricted propagator for the interior of the region is

$$\begin{aligned} g_\beta(x, y, t|x_0, y_0, 0) &= f_\beta(x, y)f_\beta(x_0, y_0) \\ &= \sum_{n=0}^{b-1} [g(r, 2n\beta + \phi, t|r_0, \phi_0, 0) \\ &\quad - g(r, 2n\beta - \phi, t|r_0, \phi_0, 0)], \end{aligned} \quad (8.1)$$

where $g(r, \phi, t|r_0, \phi_0, 0)$ is the free particle propagator in two dimensions in polar coordinates, and $f_\beta(x, y)$ is a characteristic function equal to 1 inside the wedge region and zero outside [39]. The desired class operator is now

$$C_\beta = \lim_{t'' \rightarrow \infty, t' \rightarrow -\infty} P(t'') \sum_{n=0}^{b-1} [R_n - K_n] P(t'), \quad (8.2)$$

where the projector P is $f_\beta(\hat{x}, \hat{y})$ and we have introduced the rotation operators

$$R_n = \int r dr d\phi |r, 2n\beta + \phi\rangle \langle r, \phi| \quad (8.3)$$

$$K_n = \int r dr d\phi |r, 2n\beta - \phi\rangle \langle r, \phi|. \quad (8.4)$$

Using the same method as in Eq. (6.22), it may be shown that

$$\lim_{t \rightarrow \infty} P(t) = f_\beta(\hat{p}_x, \hat{p}_y) \quad (8.5)$$

so we obtain

$$C_\beta = f_\beta(\hat{p}_x, \hat{p}_y) \sum_{n=0}^{b-1} [R_n - K_n] f_\beta(-\hat{p}_x, -\hat{p}_y). \quad (8.6)$$

Since R_n rotates by an angle $2n\beta$, we have

$$f_\beta(\hat{p}_x, \hat{p}_y)R_nf_\beta(-\hat{p}_x, -\hat{p}_y) = f_\beta(\hat{p}_x, \hat{p}_y)f_\beta(-\hat{p}_x^n, -\hat{p}_y^n)R_n^{-1}, \quad (8.7)$$

where p_x^n, p_y^n denote the momenta rotated through angle $2n\beta$. Clearly, Eq. (8.7) is zero unless $n = b/2$ (recall that b is an even integer). Similarly, it is readily shown that

$$f_\beta(\hat{p}_x, \hat{p}_y)K_nf_\beta(-\hat{p}_x, -\hat{p}_y) = 0. \quad (8.8)$$

We now have

$$C_\beta = f_\beta(\hat{p}_x, \hat{p}_y)R_{b/2}. \quad (8.9)$$

The off-diagonal term in the decoherence functional is

$$\begin{aligned} D(\bar{\beta}, \beta) &= \langle \psi | (1 - C_\beta)^\dagger C_\beta | \psi \rangle_I \\ &= \langle \psi | (R_{b/2} - 1) f(-\hat{p}_x, -\hat{p}_y) | \psi \rangle_I \end{aligned} \quad (8.10)$$

in the induced inner product, where $|\psi\rangle$ are energy eigenstates. This means that there is decoherence for states satisfying

$$R_{b/2}|\psi\rangle = |\psi\rangle. \quad (8.11)$$

This will indeed be satisfied for rotationally symmetric state and the probability then is

$$p_\beta = \langle \psi | C_\beta | \psi \rangle_I = \langle \psi | f_\beta(\hat{p}_x, \hat{p}_y) | \psi \rangle_I. \quad (8.12)$$

By symmetry, we clearly have

$$p_\beta = \frac{\beta}{2\pi} \quad (8.13)$$

as expected.

We therefore obtain the intuitively expected result and it is gratifying that a physical decoherence mechanism is not required for this model. In terms of a semiclassical interpretation, this result corresponds to a set of straight line trajectories radiating from the origin. One can think of them as paths of infinite length (and so are reparametrization invariant) coming in from infinity, bouncing off the origin, and then returning to infinity.

IX. SYSTEMS OF HARMONIC OSCILLATORS

The formalism so far concerned unbound systems, in which the (unphysical) time parameter t runs from $-\infty$ to $+\infty$. It is however very different (and simpler) for systems of harmonic oscillators, which are periodic in time. In this section we consider the case of a d -dimensional simple harmonic oscillator with Hamiltonian

$$H = \frac{1}{2}(\mathbf{p}^2 + \mathbf{x}^2). \quad (9.1)$$

(See Ref. [1] for the evolving constants analysis of this system). Much of the formalism will, however, be applicable to other systems periodic in time. Systems described by the Hamiltonian Eq. (9.1) will have period 2π so in the quantum theory we have

$$e^{iH(t+2\pi)} = e^{iHt}. \quad (9.2)$$

An important class of observables for this system are of the form

$$A = \int_0^{2\pi} dt B(t) \quad (9.3)$$

and it is easy to show, with the help of Eq. (9.2), that A commutes with H . Note that because the spectrum of H is discrete, it is not necessary to use the induced inner product.

A. Class operators

For the systems described by Hamiltonian Eq. (9.1), the natural modification of Eq. (3.11), the class operator for not entering the region Δ , is

$$C_{\bar{\Delta}} = \prod_{t=0}^{2\pi} \bar{P}(t), \quad (9.4)$$

where \bar{P} is the projector onto the region outside Δ . However, it is not hard to see that this does not in fact commute with H (unless the $P(t)$ all commute at different times). This is an operator ordering issue and is easily remedied by defining the class operator to be instead

$$C_{\bar{\Delta}} = \frac{1}{2\pi} \int_0^{2\pi} ds \prod_{t=s}^{s+2\pi} \bar{P}(t). \quad (9.5)$$

This is essentially a sum over all cyclic permutations of the operators $\bar{P}(t)$ at different times, and now commutes with the Hamiltonian.

Following steps similar to those used in previous sections, it is easily shown that

$$C_{\bar{\Delta}} = \frac{1}{2\pi} \int_0^{2\pi} ds e^{iH(s+2\pi)} g_r(s+2\pi, s) e^{-iHs}, \quad (9.6)$$

where g_r is the restricted propagator. (Note that the restricted propagator will not in general be periodic in time.) Using Eq. (9.2) together with the fact that $g_r(t, t')$ depends on time only through $(t - t')$, we have

$$C_{\bar{\Delta}} = \frac{1}{2\pi} \int_0^{2\pi} ds e^{iHs} g_r(2\pi, 0) e^{-iHs}. \quad (9.7)$$

This is of the form Eq. (9.3) so commutes with H , as expected.

The class operator $C_\Delta = 1 - C_{\bar{\Delta}}$ may be written,

$$C_\Delta = \frac{1}{2\pi} \int_0^{2\pi} ds e^{iHs} g_c(2\pi, 0) e^{-iHs}, \quad (9.8)$$

where $g_c(2\pi, 0)$ is the crossing propagator, given by a sum over paths which enter Δ at some time during the interval $[0, 2\pi]$.

B. Decoherence functional and probabilities

We may now look at the decoherence functional and the probabilities. We choose a pure initial state $|\psi\rangle$ and, in keeping with the general approach, this state is taken to be an eigenstate of the Hamiltonian

$$H|\psi\rangle = E|\psi\rangle. \quad (9.9)$$

The off-diagonal term of the decoherence functional is

$$D(\Delta, \bar{\Delta}) = \langle \psi | C_{\Delta}^{\dagger} C_{\bar{\Delta}} | \psi \rangle. \quad (9.10)$$

Inserting the explicit expressions, Eqs. (9.7) and (9.8), we have

$$D(\Delta, \bar{\Delta}) = \int_0^{2\pi} \frac{ds_1}{2\pi} \int_0^{2\pi} \frac{ds_2}{2\pi} \langle \psi | e^{iHs_1} g_c^{\dagger}(2\pi, 0) \times e^{-iH(s_1-s_2)} g_r(2\pi, 0) e^{-iHs_2} | \psi \rangle. \quad (9.11)$$

Using Eq. (9.9) this becomes

$$D(\Delta, \bar{\Delta}) = \langle \psi | g_c^{\dagger}(2\pi, 0) P_E g_r(2\pi, 0) | \psi \rangle, \quad (9.12)$$

where we have introduced the object

$$P_E = \int_0^{2\pi} \frac{ds}{2\pi} e^{-i(H-E)s}. \quad (9.13)$$

Because the spectrum of H is discrete, this is a projection operator, so satisfies $P_E^2 = P_E$.

The decoherence functional will not be diagonal in general, although one simple case in which it will is when the wave function is an eigenstate of $g_c(2\pi, 0)$ or $g_r(2\pi, 0)$. We will exhibit such states below.

When the decoherence condition is satisfied, the probabilities associated with Δ and $\bar{\Delta}$ are easily shown to be

$$p_{\Delta} = \langle \psi | g_c(2\pi, 0) | \psi \rangle, \quad p_{\bar{\Delta}} = \langle \psi | g_r(2\pi, 0) | \psi \rangle. \quad (9.14)$$

C. Some special states exhibiting approximate decoherence

We now introduce some states which exhibit approximate decoherence and quasiclassical behavior for this model. Consider first the standard coherent states of the harmonic oscillator, $|\mathbf{p}, \mathbf{x}\rangle$. They are preserved in form under unitary evolution,

$$e^{-iHt} |\mathbf{p}, \mathbf{x}\rangle = e^{-it/2} |\mathbf{p}_t, \mathbf{x}_t\rangle, \quad (9.15)$$

where $\mathbf{p}_t, \mathbf{x}_t$ are the classical solutions matching \mathbf{p}, \mathbf{x} at $t = 0$, hence they are strongly peaked about the classical path. There is a set of states which are natural analogues of these states for the timeless models considered here. They were referred to in Ref. [40] as “timeless coherent states” and are defined by

$$\begin{aligned} |\phi_{\mathbf{p}\mathbf{x}}\rangle &= P_E |\mathbf{p}, \mathbf{x}\rangle = \int_0^{2\pi} \frac{dt}{2\pi} e^{-i(H-E)t} |\mathbf{p}, \mathbf{x}\rangle \\ &= \int_0^{2\pi} \frac{dt}{2\pi} e^{i(E-\frac{1}{2})t} |\mathbf{p}_t, \mathbf{x}_t\rangle. \end{aligned} \quad (9.16)$$

They are clearly eigenstates of H with eigenvalue E and are concentrated around the entire classical path with initial data \mathbf{p}, \mathbf{x} . They are not normalized to 1 exactly, but if the initial data satisfies $E = \frac{1}{2}(\mathbf{p}^2 + \mathbf{x}^2)$ then the coherent states $|\mathbf{p}, \mathbf{x}\rangle$ are approximate eigenstates of P_E and the timeless coherent states are then approximately normalized to 1. Further properties of these states are described in Ref. [40].

Now consider a timeless coherent state whose trajectory $\mathbf{p}_t, \mathbf{x}_t$ lies entirely within the region $\bar{\Delta}$. This region could, for example, be a large rectangular region in configuration space. Or it could be a tube following the classical trajectory but broadened out beyond the scale of quantum fluctuations. In both these cases, if \bar{P} is the projector onto the region $\bar{\Delta}$, then we clearly have

$$\bar{P} |\phi_{\mathbf{p}\mathbf{x}}\rangle \approx |\phi_{\mathbf{p}\mathbf{x}}\rangle \quad (9.17)$$

and

$$P |\phi_{\mathbf{p}\mathbf{x}}\rangle \approx 0. \quad (9.18)$$

We assert that with this choice of $\bar{\Delta}$, the state $|\phi_{\mathbf{p}\mathbf{x}}\rangle$ will give approximate decoherence. There are two ways to see this.

First, from the (informal) expression Eq. (9.5), the result Eq. (9.17) together with the fact that the state is an eigenstate of H imply that it is also an approximate eigenstate of $\bar{P}(t)$, so will be an approximate eigenstates of the class operator. This means there is approximate decoherence.

Second, and perhaps a little more rigorously, we use the expression for the decoherence functional Eq. (9.12). The important thing is to consider the action of the restricted propagator $g_r(2\pi, 0)$ on the state $|\phi_{\mathbf{p}\mathbf{x}}\rangle$, which, via Eq. (9.16), boils down to its action on the coherent state $|\mathbf{p}, \mathbf{x}\rangle$. Restricted propagators are very difficult to calculate for arbitrary regions, but their path integral form gives an intuitive picture of their properties. It is

$$g_r(\mathbf{x}'', 2\pi | \mathbf{x}', 0) = \int_{\bar{\Delta}} \mathcal{D}\mathbf{x} \exp\left(i \int_0^{2\pi} dt \left[\frac{1}{2} \dot{\mathbf{x}}^2 - \frac{1}{2} \mathbf{x}^2 \right]\right). \quad (9.19)$$

This is a sum over paths $\mathbf{x}(t)$ which remain always in the region $\bar{\Delta}$ and satisfying the endpoint conditions $\mathbf{x}(0) = \mathbf{x}'$, $\mathbf{x}(2\pi) = \mathbf{x}''$. Suppose that the trajectory $\mathbf{p}_t, \mathbf{x}_t$ of the coherent state $|\mathbf{p}, \mathbf{x}\rangle$ remains entirely within the region $\bar{\Delta}$ (and does not approach the boundary). Then, when this initial state is attached to the restricted propagator, the path integral will be dominated by the classical path with initial data \mathbf{p}, \mathbf{x} . The path integral will therefore be approximately the same as the unrestricted path integral, which means that

$$g_r(2\pi, 0)|\mathbf{p}, \mathbf{x}\rangle \approx |\mathbf{p}, \mathbf{x}\rangle. \quad (9.20)$$

It follows that there will be approximate decoherence and the probability for finding the particle in the region $\bar{\Delta}$ is approximately 1.

So for these specially chosen regions that entirely contain the trajectory of the timeless coherent state we get approximate decoherence and the expected probabilities. Note that these heuristic arguments only work for periodic systems in which certain states remain coherent. For the systems considered in earlier sections involving an infinite range of time, the spreading of wave packets would render such heuristic arguments invalid.

For most other choices of Δ , however, there is no decoherence and probabilities cannot be assigned without a decoherence mechanism. This will be pursued elsewhere [36].

X. SUMMARY AND CONCLUSIONS

We have discussed the issues involved in defining class operators for the decoherent histories analysis of reparametrization-invariant systems and made a specific proposal for such operators. The class operators defined are based on certain reasonable classical expressions and reduce to projection operators when everything commutes. They commute with the Hamiltonian so fully respect reparametrization invariance. They do not, however, exhibit the localization properties of their classical counterparts. This is because there is an incompatibility between localization and the constraint and in our definition we have made the choice that the constraint should take precedence.

We compared with the evolving constants approach and noted that the difference between that approach and the present one concerned the different ways in which equivalent classical expressions are turned into quantum operators.

We showed that our class operators gave the correct and expected results when applied to standard nonrelativistic quantum mechanics written in parametrized form. These results also showed how spacetime questions in nonrelativistic quantum mechanics can be expressed in a fully spacetime form, involving an inner product defined on spacetime.

We applied our formalism to some simple examples involving the free particle in one and two dimensions. These examples showed that the class operators could be easily calculated. Furthermore, there were some situations in which decoherence was possible for special states, without the need for an environment. However, the results were not always easy to interpret, and this is largely due to the fact that the classical limit of the quantum theory brings in reflecting potentials.

We briefly discussed the relativistic particle and considered the question of the probability of crossing a spacelike surface. We found that only states antisymmetric about the

surface give decoherence and the crossing probability is then zero. This means that our formalism does not appear to reproduce earlier heuristic formulas for surface crossings. Furthermore, this result also emphasized that our approach is in fact a genuinely different proposal for the class operators compared to other approaches, and not a formalization of earlier more heuristic ideas.

The formalism boiled down to particularly simple expressions for the case of systems of noninteracting harmonic oscillators and we exhibited some simple eigenstates which gave approximate decoherence and which had a clear semiclassical interpretation.

In terms of applications, we have largely concentrated in this paper on simple quantum-mechanical examples. Future papers will address the quantization of cosmological models described by a Wheeler-DeWitt equation. These are considerably more complicated than the models considered here for two reasons. First of all, the Wheeler-DeWitt equation for most cosmological models has a nontrivial potential (and often also a nontrivial metric). This makes the restricted propagators very difficult to calculate—the models considered in this paper relied heavily on the method of images which requires a high degree of symmetry. Second, decoherence of histories for general initial states usually only occurs when there is a physical mechanism for decoherence. This usually means coupling to an environment, which is a significant complication. These issues will be addressed in a future paper [36].

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APPENDIX: A USEFUL MATHEMATICAL RESULT

Here we give a simple and useful result for simplifying induced inner product expressions of the form Eq. (5.12). The expressions below specifically refer to the parametrized nonrelativistic particle with Hamiltonian constraint Eq. (5.2)—they are not valid for parametrized systems which are quadratic in all the momenta.

The decoherence functional is an expression of the general form

$$\langle \Psi_{\lambda'} | A | \Psi_{\lambda} \rangle, \quad (A1)$$

where A commutes with the constraint, and it is useful to show how this expression reduces to a simpler expression on the original Hilbert space \mathcal{H}_x . In all the expressions we are interested in, A will commute with \hat{t} , so A has the form

$$A = \int_{-\infty}^{\infty} dt B(t) \otimes |t\rangle\langle t|, \quad (A2)$$

where $B(t)$ acts on \mathcal{H}_x only. Using the fact that $[A, H] = 0$, it is straightforward to deduce that

$$B(t) = e^{-iht}B(0)e^{iht} \quad (\text{A3})$$

(where note that the signs in the exponents are not the ones associated with Heisenberg picture evolution). We now have

$$\begin{aligned} \langle \Psi_{\lambda'} | A | \Psi_{\lambda} \rangle &= \int dt \int dx' dx \Psi_{\lambda'}^*(x', t) \langle x' | B(t) | x \rangle \Psi_{\lambda}(x, t) \\ &= \frac{1}{2\pi} \int dt e^{-i\lambda t + i\lambda' t} \int dx' dx \psi^*(x', t) \\ &\quad \times \langle x' | e^{-iht} B(0) e^{iht} | x \rangle \psi(x, t). \end{aligned} \quad (\text{A4})$$

Now noting that

$$\int dx e^{iht} |x\rangle \psi(x, t) = \int dx |x\rangle \psi(x, 0) = |\psi\rangle \quad (\text{A5})$$

we finally obtain

$$\langle \Psi_{\lambda'} | A | \Psi_{\lambda} \rangle = \delta(\lambda - \lambda') \langle \psi | B(0) | \psi \rangle. \quad (\text{A6})$$

The expression on the left is in terms of the auxiliary inner product on $\mathcal{H}_x \otimes \mathcal{H}_t$. The inner product on the right is the usual one on \mathcal{H}_x . Hence expressions of the form Eq. (A1) are readily evaluated once one has read off $B(t)$ in Eq. (A3).

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